A ring structure on $\mathscr{Z}_0(\mathbb{C}^4)$ and an inverse twistor function formula

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Abstract. *The well-known Penrose's integral formula allows to express each free spinorial field verifying the Dirac equation by a «twistorial function».*
The present paper gives conversely an explicit way to compute a twistorial function associated to a field.

INTRODUCTION

The conformal structure inherited by an open subset U from the complexified Minkowski space can be reconduced to the complex structure of a corrispondent open subset U'' of $\mathbb{P}^3$ (the projective twistor space).

Therefore every object or property invariant by conformal transformations must be expressable in terms of the complex structure of the open subsets of $\mathbb{P}^3$.

It is an essential part of this program the proof given by R. Penrose that every free spinorial field $\{\varphi_{p_0 p_1}(x)\}_{p_0+p_1=0}$ with zero-rest-mass and helicity $n/2$ verifying the Dirac equation on U corresponds to a cohomology class in the space $H^1(U'', \mathcal{O}(-n-2))$. For example if U is all the affine complex Minkowski space and f is a «twistorial function» representing a cohomology class the corresponding field is given by the formula:

$$\varphi_{p_0 p_1}(x) = \frac{1}{2\pi i} \cdot \oint \pi_0^{p_0} \cdot \pi_1^{p_1} \cdot f(i x_{00}\pi_0 + i x_{01}\pi_1, i x_{10}\pi_0 + i x_{11}\pi_1, \pi_0, \pi_1)\, \Delta\pi .$$

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The function f keeps memory of each intrinsic property of the field, that is every conformal property of the field can be obtained in a pure geometric way by the function f.

The present article gives an explicit way of computing a twistorial function associated to a field. The formula is obtained by using sistematically a ring structure defined on the space $\mathscr{X}_0(\mathbb{C}^4)$ of holomorphic massless free scalar fields on the affine Minkowski $\mathbb{C}^4$ and then considering the spaces $\mathscr{X}_n(\mathbb{C}^4)$ of vectorial fields as modules on $\mathscr{X}_0(\mathbb{C}^4)$.

These algebraic structures make these spaces respectively isomorphic to the ring $\mathscr{O}(C)$ of holomorphic functions on the cone $C=\{z\in\mathbb{C}^4 : z_{00}\cdot z_{11}-z_{01}\cdot z_{10}=0\}$ and to the modules $\mathscr{T}^n_S(C)$ of holomorphic functions on C vanishing with order at least n on a plane S in C.

§ 1. THE SPACE $\mathscr{X}_0$ OF MASSLESS FREE SCALAR FIELDS

For every $l=(l_{00},l_{01},l_{10},l_{11})$ in $\mathbb{N}^4$ we set: $l_0'=l_{00}+l_{01}$, $l_0''=l_{00}+l_{10}$, $l_1'=l_{10}+l_{11}$, $l_1''=l_{01}+l_{11}$, $|l|=l_{00}+l_{01}+l_{10}+l_{11}$, $M=\{m=(m_0',m_1',m_0'',m_1'')\in\mathbb{N}^4 : m_0'+m_1'=m_0''+m_1''\}$, $s(l)=(l_0',l_1',l_0'',l_1'')$.

We say that l, r in $\mathbb{N}^4$ are equivalent ($l\sim r$) if $s(l)=s(r)$, observe that if $s(l)=m$ then $s^{-1}(m)=\{l+(-b,b,b,-b) : -\min(l_{01},l_{10})\leqslant b\leqslant\min(l_{00},l_{11})\}$. M represents the quotient set $\mathbb{N}^4/\sim$.

For every l in $\mathbb{N}^4$ (with $m=s(l)$) we set: $B(l)=\binom{l_0'}{l_{00}}\cdot\binom{l_1'}{l_{11}}$ and $C(l)=C(m)=\binom{l_0''+l_1''}{l_0''}$. Developing the equation $(1+x)^{m_0''+m_1''}=(1+x)^{m_0'}\cdot(1+x)^{m_1'}$ we have: $C(m)=\sum_{s(l)=m}B(l)$. Set $N(l)=N(m)=\#[l]=\min(l_{00},l_{11})+\min(l_{01},l_{10})+1$ observe also that $N(l)\leqslant|l|+1$ and $N(m)\leqslant C(m)\leqslant 2^{|m|/2}$.

For a general x in the cone: $C=\{x\in\mathbb{C}^4 : x_{00}\cdot x_{11}-x_{01}\cdot x_{10}=0\}$ it holds: $\dfrac{x_{01}\cdot x_{10}}{x_{00}\cdot x_{11}}=1$, therefore we have $x^l=x^r$ for $l\sim r$.

For every m in M and every x in C we can set: $x^{(m)}=x^l$ whenever $m=s(l)$.

For every m in M we define the elementary polynomial:

$$Q_m(x)=\sum_{s(l)=m}B(l)\cdot x^l.$$

The polynomial Q_m is homogeneous of degree $|m|/2$ and takes origin as the Penrose trasform in the affine Minkowski space of the twistorial function $(\omega_0^{m_0'}\cdot\omega_1^{m_1'})/(\pi_0^{m_0''+1}\cdot\pi_1^{m_1''+1})$ since:

$$\mathscr{P}_0(\omega_0^{m_0'} \cdot \omega_1^{m_1'}/(\pi_0^{m_0''+1} \cdot \pi_1^{m_1''+1}))(x) = i^{|m|/2} \cdot Q_m(x)$$

(where:

$$\mathscr{P}_0(f)(x) = \frac{1}{2\pi i}\oint f(i\,x_{00}\pi_0 + i\,x_{01}\pi_1, i\,x_{10}\pi_0 + i\,x_{11}\pi_1, \pi_0, \pi_1) \cdot (\pi_0\lambda\pi_1 - \pi_1\lambda\pi_0)$$

for $f \in \mathscr{O}(-2)\,(\mathbb{C}^4_{\omega,\pi} - (H_{\pi_0} \cup H_{\pi_1})))$.

We will consider $Q_m = 0$ if m is not positive.

For every m in M and every x in C we have:

$$Q_m(x) = C(m) \cdot x^{(m)}.$$

Denoted by $\Box$ the operator $\Box = \dfrac{\partial^2}{\partial x_{00}\,\partial x_{11}} - \dfrac{\partial^2}{\partial x_{01}\,\partial x_{10}}$ we set:

$$\mathscr{Z}_0(\mathbb{C}^4) = \{\varphi \in \mathscr{O}(\mathbb{C}^4) : \Box\varphi = 0\}.$$

THEOREM (1.1). *A function φ of $\mathscr{O}(\mathbb{C}^4)$ is in $\mathscr{Z}_0(\mathbb{C}^4)$ if and only if can be expressed as a series:*

$$\varphi(x) = \sum_{m \in M} a_m \cdot Q_m(x).$$

In this case the expression is unique and

$$\lim_{d\to\infty} \sqrt[d]{|\varphi|^{(d)}} = \lim_{d\to\infty} \sqrt[d]{|a|^{(d)}} = \lim_{d\to\infty} \sqrt[d]{\sum_{|l|=d} |a_l|} = 0.$$

Proof. In one of the two direction the theorem holds for the continuity of the operator $\Box$ and for:

$$\frac{\partial Q_m}{\partial x_{00}} = m_0' \cdot Q_{m-(1,0,1,0)}, \quad \frac{\partial Q_m}{\partial x_{01}} = m_0' \cdot Q_{m-(1,0,0,1)}$$

$$\frac{\partial Q_m}{\partial x_{10}} = m_1' \cdot Q_{m-(0,1,1,0)}, \quad \frac{\partial Q_m}{\partial x_{11}} = m_1' \cdot Q_{m-(0,1,0,1)}$$

implying $\Box\, Q_m = 0$.

In the other direction taken $\varphi \in \mathscr{Z}_0\,\mathbb{C}^4$ and written

$$\varphi = \sum_{d \geq 0} \varphi^{(d)}$$

with $\varphi^{(d)}$ homogeneous of degree d (and still $\square\, \varphi^{(d)} = 0$) the theorem follows fram this fact: the set $B_d = \{Q_m : |m| = 2d\}$ is a base for the space

$$\mathscr{Y}_0^{(d)} = \{\varphi \in \mathscr{Y}_0(\mathbb{C}^4) : \varphi \text{ homogeneous of degree } d\}.$$

Infact from the exact sequence:

$$0 \to \mathscr{Y}_0^{(d)} \to \mathcal{O}^{(d)} \xrightarrow{\square} \mathcal{O}^{(d-2)} \to 0$$

follows $\dim \mathscr{Y}_0^{(d)} = (d+1)^2$, moreover the polynomials Q_m of B_d are exactly $(d+1)^2$ and indipendent, since: if $\sum_{|m|=2d} a_m \cdot Q_m = 0$ then $\sum_{|l|=d} B(l) \cdot a_{s(l)} \cdot x^l = 0$ and therefore $a_{s(l)} = 0$ for every l.

This proves also the unicity of the expression.

The limit is proved remembering that a power series $\sum_l a_l \cdot x^l$ converges in all $\mathbb{C}^4$ if and only if $\lim_{d\to\infty} \sqrt[d]{|a|^{(d)}} = 0$ and observing that for $\varphi(x) = \sum_l B(l) \cdot a_{s(l)} \cdot x^l$ it holds: $|\varphi|^{(2d)} = \sum_{|m|=2d} |a_m| \cdot C(m) \leqslant 2^d \cdot \sum_{|m|=2d} |a_m| = 2^d \cdot |a|^{(2d)}$. ∎

THEOREM (1.2). *The restriction map $\rho : \mathscr{Y}_0(\mathbb{C}^4) \to \mathcal{O}(C)$ is an isomorphism of vector spaces.*

Proof. If $\varphi = \sum_k a_k \cdot x^k$ has restriction null on C then $\varphi = (x_{00} \cdot x_{11} - x_{01} \cdot x_{10}) \cdot \psi$ with $\psi = \sum_h b_h \cdot x^h$. For every $r \in \mathbb{N}^4$ let's consider the linear map $\gamma_r : \mathcal{O}(\mathbb{C}^4) \to \mathbb{C}$ defined by $\gamma_r(\sum_k a_k \cdot x^k) = \sum_{k\sim r} a_k$.

Since $h + (1,0,0,1) \sim h + (0,1,1,0)$ then $\sum_{k\sim r} a_k = \gamma_r(\varphi) = \gamma_r(\sum_h b_h \cdot x^{h+(1,0,0,1)} - \sum_h b_h \cdot x^{h+(0,1,1,0)}) = \sum_{|h|=|x|-2} b_h \cdot \gamma_r(x^{h+(1,0,0,1)} - x^{h+(0,1,1,0)}) = 0$ for every r.

For a function $\varphi = \sum_m \alpha_m \cdot Q_m(x) = \sum_k \alpha_{s(k)} \cdot B(k) \cdot x^k$ in $\mathscr{Y}_0(\mathbb{C}^4)$ it follows:

$$0 = \sum_{k\sim r} \alpha_{s(k)} \cdot B(k) = \alpha_{s(r)} \cdot \sum_{k\sim r} B(k) = \alpha_{s(r)} \cdot C(r)$$

therefore $\alpha_m = 0$ for every $m \in M$ and then $\varphi = 0$. This proves the map ρ is injective.

Let's prove now that ρ is surjective, taken $\psi = \sum a_h \cdot x^h$ in $\mathcal{O}(\mathbb{C}^4)$ let's consider the function $\varphi = \sum_{m\in M} \frac{\alpha_m}{C(m)} \cdot Q_m(x)$ where $\alpha_m = \sum_{s(h)=m} a_h$, since

$$\lim_{2\cdot d\to\infty} \sqrt[2\cdot d]{\sum_{|m|=2\cdot d} \frac{|\alpha_m|}{C(m)}} \leqslant \lim_{2\cdot d\to\infty} \sqrt[2\cdot d]{\sum_{|h|=d} |a_h|} = 0$$

the function φ is entire and

$$\rho(\varphi) = \sum_{m \in M} \frac{\alpha_m}{C(m)} \cdot C(m) \cdot x^{(m)} = \Sigma\, \alpha_m \cdot x^{(m)} =$$

$$= \sum_m \left(\sum_{s(h)=m} a_h \right) \cdot x^{(m)} = \sum_h a_h \cdot x^h |_C = \psi |_C \qquad \blacksquare$$

Observe that taken $\varphi = \sum_m a_m \cdot Q_m$ then for every $r > 0$ and $m \in M$ it holds:

$$|a_m| \leqslant \frac{\|\varphi\|_r}{r^{|m|/2}}.$$

Infact $\varphi = \sum_l B(l) \cdot a_{s(l)} \cdot x^l$ therefore

$$B(l) \cdot |a_{s(l)}| \leqslant \frac{\|\varphi\|_r}{r^{|l|}}, \text{ then } \sum_{s(l)=m} B(l) \cdot |a_{s(l)}| \leqslant$$

$$\leqslant N(m) \cdot \frac{\|\varphi\|_r}{r^{|m|/2}}, \text{ that is } C(m) \cdot |a_m| \leqslant \frac{\|\varphi\|_r}{r^{|m|/2}} \cdot N(m).$$

§2. THE FRECHET ALGEBRA $\mathscr{X}_0(\mathbb{C}^4)$

We define on $\mathcal{O}(\mathbb{C}^4)$ a product $*$ such that $Q_m * Q_p = Q_{m+p}$ whenever $m \geqslant 0$ and $p \geqslant 0$.

For every $\varphi(x) = \Sigma\, a_h \cdot x^h$ and $\psi(x) = \Sigma\, b_k \cdot x^k$ in $\mathcal{O}(\mathbb{C}^4)$ we define:

$$(\varphi * \psi)(x) = \sum_l \left[\sum_{h+k \sim l} \frac{1}{N(h) \cdot N(k)} \cdot \frac{B(l)}{B(h) \cdot B(k)} \cdot a_h \cdot b_k \right] \cdot x^l.$$

The series $\varphi * \psi$ is entire, infact for every $R > 0$ the series $\Sigma\, |a_h| \cdot (A \cdot R)^{|h|}$, $\Sigma\, |b_k| \cdot (A \cdot R)^{|k|}$ and $\sum_u \sum_{h+k=u} |a_h| \cdot |b_k| \cdot (A \cdot R)^{|u|}$ are convergent for the absolute convergence of φ, ψ.

Therefore for $\|x\| \leqslant R$:

$$\sum_l |\alpha_l(\varphi * \psi)| \cdot |x^l| \leqslant \sum_u N(u) \cdot \sum_{h+k=u} |a_h| \cdot |b_k| \cdot (2 \cdot R)^{|u|} \leqslant$$

$$\leqslant \sum_u \sum_{h+k=u} |a_h| \cdot |b_k| \cdot (4 \cdot R)^{|u|} < +\infty.$$

THEOREM (2.1). *$(\mathscr{X}_0(\mathbb{C}^4), *)$ is a Fréchet algebra with 1 as a unit element.*

Proof. The product $*: \mathcal{O}(\mathbb{C}^4) \times \mathcal{O}(\mathbb{C}^4) \to \mathcal{O}(C^4)$ is continous for the uniform convergence topology.

There exists a constant K such that $\sum_{|l|=d} \sum_{h+k\sim l} 1 \leqslant K^d$, taken $A > 2K$ we have:

$$|a_h| \leqslant \frac{\|\varphi\|_{A\cdot R}}{(A\cdot R)^{|h|}} \quad \text{and} \quad |b_k| \leqslant \frac{\|\psi\|_{A\cdot R}}{(A\cdot R)^{|k|}}$$

therefore for $\|x\| \leqslant R$:

$$|(\varphi * \psi)(x)| \leqslant \sum_l \sum_{h+k\sim l} B(l)\cdot|a_h|\cdot|b_k|\cdot R^{|l|} \leqslant$$

$$\leqslant \|\varphi\|_{A\cdot R}\cdot\|\psi\|_{A\cdot R}\cdot\Sigma\left(\frac{2K}{A}\right)^d.$$

The commutativity and the distributivity of the product follow almost immediately from the definition of $*$.

Let's prove the associativity, let $\varphi = \Sigma\, a_h \cdot x^h$, $\psi = \Sigma\, b_k \cdot x^k$ and $\vartheta = \Sigma\, c_j \cdot x^j$.

For $(\varphi * \psi) * \vartheta = \Sigma\, \alpha_p \cdot x^p$ we have

$$\alpha_p = \sum_{l+j\sim p} \frac{1}{N(j)N(l)}\cdot\frac{B(p)}{B(l)B(j)}\cdot c_j\cdot\sum_{h+k\sim l}\frac{1}{N(h)N(k)}\cdot\frac{B(l)}{B(h)B(k)}\cdot a_h\cdot b_k.$$

Let $A = \{(l,j,h,k): l+j\sim p,\ h+k\sim l\}$

$A' = \{(j,h,k): \exists l \text{ such that } (l,j,h,k)\in A\}$

and for every (j,h,k) in A':

$$F(j,h,k) = \{(l,j,h,k)\in A\}.$$

It holds:

$$A' = \{(j,h,k): j+h+k\sim p\}$$

and

$$F(j,h,k) = \{(l,j,h,k): l\sim h+k\}.$$

Therefore:

$$\alpha_p = B(p)\cdot\sum_{(l,j,h,k)\in A}\frac{1}{N(l)}\cdot\frac{1}{N(h)N(k)N(j)}\cdot\frac{1}{B(h)B(k)B(j)}\cdot a_h\cdot b_k\cdot c_j =$$

$$= B(p) \cdot \sum_{(j,h,k)\in A'} \frac{1}{N(h)N(k)N(j)} \cdot \frac{1}{B(h)B(k)B(j)} \cdot a_h \cdot b_k \cdot c_j$$

the same holds for the coefficient of place p in the expansion of $\varphi * (\psi * \vartheta)$.

Set $x_{jl}^{*0} = 1$, $x_{jl}^{*u} = x_{jl} * \ldots * x_{jl}$ (u times) for $u \geqslant 1$ and $x^{*l} = x_{00}^{*l_{00}} * x_{01}^{*l_{01}} * * x_{10}^{*l_{10}} * x_{11}^{*l_{11}}$ for $l = (l_{00}, l_{01}, l_{10}, l_{11})$.

Since $x_{00} * Q_m = Q_{m+(1,0,1,0)}$, $x_{01} * Q_m = Q_{m+(1,0,0,1)}$, $x_{10} * Q_m = Q_{m+(0,1,1,0)}$ and $x_{11} * Q_m = Q_{m+(0,1,0,1)}$ proceeding by induction an $|l|$ it is possible to prove that:

$$x^{*l} = Q_{s(l)}.$$

Therefore $Q_m * Q_p = Q_{s(l)} * Q_{s(h)} = x^{*l} * x^{*h} = x^{*(l+h)} = Q_{m+p}$.

Taken $\varphi = \sum_m a_m \cdot Q_m$ and $\psi = \sum_p b_p \cdot Q_p$ in $\mathscr{X}_0(\mathbb{C}^4)$ for the continuity of $*$ it holds: $\varphi * \psi = \sum_q (\sum_{m+p=q} a_m \cdot b_p) \cdot Q_q$ therefore $\mathscr{X}_0(\mathbb{C}^4)$ is a closed subring of $\mathcal{O}(\mathbb{C}^4)$. At last for every $\varphi = \sum_m a_m \cdot Q_m = \sum_k B(k) \cdot a_{s(k)} \cdot x^k$ we have:

$$(1 * \varphi)(x) = \sum_l \left[\sum_{k\sim l} \frac{1}{1 \cdot N(k)} \cdot \frac{1}{1 \cdot B(k)} \cdot B(k) \cdot a_{s(k)}\right] x^l =$$

$$= \sum_l B(l) \cdot a_{s(l)} \cdot x^l \cdot \left(\sum_{k\sim l} \frac{1}{N(l)}\right) = \varphi(x). \qquad \blacksquare$$

§3. AN INVERSE TWISTOR FORMULA FOR SCALAR FIELDS

THEOREM (3.1). *The map $\vartheta : \mathcal{O}(C) \to \mathscr{X}_0(\mathbb{C}^4)$ defined by:*

$$\vartheta\left(\left[\sum_h a_h \cdot z^h\right]\right) = \sum_h a_h \cdot Q_{s(h)}$$

is well defined and a topological ring isomorphism.

Proof. We already observed that the series $\sum_h a_h \cdot Q_{s(h)}$ is covergent on $\mathbb{C}^4$ if $\sum_h a_h \cdot z^h$ is entire, therefore the series $\sum_h a_h \cdot Q_{s(h)}$ is a function of $\mathscr{X}_0(\mathbb{C}^4)$.

Since $\vartheta[(z_{00}z_{11} - z_{01}z_{10}) \cdot z^h] = 0$, $\vartheta[k] = 0$ for every function $k \in \mathcal{O}(\mathbb{C}^4)$ vanishing on C and ϑ is a well defined ring homomorphism.

The map ϑ is injective: if $\vartheta[k] = 0$ then $\sum_{s(h)=m} a_h = 0$ for every $m \in M$, then for a general z in C:

$$k(z)=\sum_{\substack{m\\ r\in s^{-1}(m)}}\sum_{b=-\min(r_{01},r_{10})}^{\min(r_{00},r_{11})} a_{r+(-b,b,b,-b)}\cdot z^r\cdot\left(\frac{z_{01}\cdot z_{10}}{z_{00}\cdot z_{11}}\right)^b=$$

$$=\sum_m\left(\sum_{s(h)=m} a_h\right)\cdot z^{(m)}=0.$$

Taken $\varphi=\sum_m a_m\cdot Q_m$ the function $k(z)=\sum_k\frac{a_{s(h)}}{N(h)}\cdot z^h$ is entire with $\vartheta[k]=\varphi$, therefore the map ϑ is surjective.

The inverse map $\tau=\vartheta^{-1}:\mathscr{X}_0(\mathbf{C}^4)\to\mathcal{O}(C)$ is continuous (and therefore a topological isomorphism among Fréchet spaces). Infact taken two numbers A, $K>0$ such that $\sum_{m\in M}\frac{1}{A^{|m|/2}}\leqslant K$ we can prove that $\|\tau(\varphi)\|_r\leqslant K\cdot\|\varphi\|_{A\cdot r}$ for every $r>0$: for $z\in C$ with $\|z\|\leqslant r$, we have

$$\|\tau(\varphi)\|_r\leqslant\sum_m|a_m|\cdot|z^{(m)}|\leqslant\sum_m\frac{\|\varphi\|_{A\cdot r}}{(A\cdot r)^{|m|/2}}\cdot r^{|m|/2}\leqslant$$

$$\leqslant\|\varphi\|_{A\cdot r}\cdot K.$$ ■

Let's denote by $\tau:\mathscr{X}_0(\mathbf{C}^4)\to\mathcal{O}(C)$ the inverse map $\tau=\vartheta^{-1}$.

Since C is the space of maximal ideals of $\mathscr{X}_0$ with the right topology, the map τ can be considered as a «Fourier transform» of the Fréchet algebra $(\mathscr{X}_0, *)$.

THEOREM (3.2). *For every $\varphi\in\mathscr{X}_0(\mathbf{C}^4)$ and $z\in C$ it holds:*

$$\tau(\varphi)(z)=\frac{1}{(2\pi\mathrm{i})^2}\cdot\oint\!\!\oint\frac{1}{\lambda_0\cdot\lambda_1}\cdot\varphi\left(\frac{z_{00}}{\lambda_0},\frac{z_{01}}{\lambda_1},\frac{z_{10}}{\lambda_0},\frac{z_{11}}{\lambda_1}\right)\cdot\chi(\lambda_0,\lambda_1)\cdot d\lambda$$

where the integrals are taken on small cicles around 0 in $\mathbf{C}_{\lambda_0}$ and $\mathbf{C}_{\lambda_1}$, and $\chi(y)$ is the function:

$$\chi(y_0,y_1)=\sum_{j_0,j_1}\frac{1}{\binom{j_0+j_1}{j_0}}\cdot y_0^{j_0}\cdot y_1^{j_1}$$

(holomorphic in a neighborhood of zero in $\mathbf{C}^2$).

Proof. Since $\mathscr{X}_0(\mathbf{C}^4)$ is «generated» by the polynomials Q_m, for the continuity of τ and of the integral is enough to prove the formula for the polynomials Q_m; that is the formula gives $z^{(m)}$ when $\varphi = Q_m$.

In fact:

$$\frac{1}{(2\pi 1)^2} \cdot \oint\oint \frac{1}{\lambda_0 \lambda_1} \cdot \left[\sum_{s(l)=m} B(l) \cdot \frac{z^l}{\lambda_0^{l_0''} \cdot \lambda_1^{l_1''}} \right] \cdot \left[\sum_{j_0, j_1} \frac{1}{\binom{j_0 + j_1}{j_0}} \cdot \lambda_0^{j_0} \cdot \lambda_1^{j_1} \right] \cdot d\lambda =$$

$$= \sum_{s(l)=m} \sum_{j} \frac{1}{\binom{j_0 + j_1}{j_0}} \cdot B(l) \cdot z^l \cdot \frac{1}{(2\pi 1)^2} \oint\oint \lambda_0^{j_0 - l_0'' - 1} \cdot \lambda_1^{j_1 - l_1'' - 1} \cdot d\lambda_0 \cdot d\lambda_1 =$$

$$= \sum_{s(l)=m} B(l) \cdot \frac{1}{\binom{l_0'' + l_1''}{l_0''}} \cdot z^l = z^{(m)}. \qquad \blacksquare$$

With an easy computation is possible to prove that: if $\varphi(x) = \varphi(0, x_{01}, 0, x_{11})$ or $\varphi(x) = \varphi(x_{00}, 0, x_{10}, 0)$ then $\tau(\varphi) = \varphi$.

In $\mathbb{P}^3$ let $L = \{[\omega_0, \omega_1, \pi_0, \pi_1] : \pi_0 = \pi_1 = 0\}$, if we consider the open subsets $U_j = \{[\omega, \pi] : \pi_j \neq 0\}$ for $j = 0, 1$ we get an open covering $\mathcal{U} = \{U_0, U_1\}$ of $\mathbb{P}^3 - - L$ such that:

$$H^1(\mathcal{U}, \mathcal{O}(k)) = H^1(\mathbb{P}^3 - L, \mathcal{O}(k))$$

for every $k \in \mathbb{Z}$.

To have a cohomology class on $\mathbb{P}^3 - L$ is then sufficient to furnish a section f holomorphic on $U_0 \cap U_1$ of $\mathcal{O}(k)$; that is a function f on $\mathbb{C}^2_\omega \times \mathbb{C}^{*2}_\pi$ holomorphic and homogeneous of degree k.

We will need in the following the map:

$$h_0 : \mathcal{O}(C) \to H^1(\mathcal{U}, \mathcal{O}(-2))$$

defined by:

$$h_0(k)([\omega, \pi]) = \left[\frac{1}{\pi_0 \cdot \pi_1} \cdot k\left(\frac{-i \cdot \omega_0}{\pi_0}, \frac{-i \cdot \omega_0}{\pi_1}, \frac{-i \cdot \omega_1}{\pi_0}, \frac{-i \cdot \omega_1}{\pi_1} \right) \right].$$

Set $\widetilde{U}_j = \{(\omega, \pi) \in \mathbb{C}^4 : \pi_j \neq 0\}$ for $j = 0, 1$ and $\widetilde{U} = \widetilde{U}_0 \cap \widetilde{U}_1$. Proceeding as in $[F]$ we define the projection operator (for every $m \in \mathbb{Z}$):

$$p : \mathcal{O}(m)(\widetilde{U}) \to \mathcal{O}(m)(\widetilde{U})$$

by

$$(pf)(\omega, \pi) = \frac{1}{(2\pi i)^2} \cdot \oint_{|t_1|=R_1} \oint_{|t_2|=R_2} \frac{f(\omega, t)}{(t_0 - \pi_0)(t_1 - \pi_1)} \, dt_0 \, dt_1$$

for every $|\pi_0| > R_0 > 0$ and $|\pi_1| > R_1 > 0$; and the subspace:

$$F_m = \{ f \in \mathcal{O}(m)(\widetilde{U}) : pf = f \}.$$

As in [F] it is possible to prove that:

$$F_m = \left\{ f \in \mathcal{O}(m)(\widetilde{U}) : f(\omega, \pi) = \sum_{\substack{k,l \\ |k|-|l|=m+2}} a_{kl} \cdot \frac{\omega_0^{k_0} \omega_1^{k_1}}{\pi_0^{l_0+1} \pi_1^{l_1+1}} \right\} =$$

$$= \{ f \in \mathcal{O}(m)(\widetilde{U}) : \lim_{\pi_0 \to \infty} f(\omega, \pi) = 0 \text{ for every } \omega \text{ and } \pi_1 \neq 0$$

$$\text{and } \lim_{\pi_1 \to \infty} f(\omega, \pi) = 0 \text{ for every } \omega \text{ and } \pi_0 \neq 0 \}.$$

The inclusion $j : F_m \hookrightarrow \mathcal{O}(m)(\widetilde{U})$ induces a topological isomorphism:

$$\hat{j} : F_m \to H^1(\mathcal{U}, \mathcal{O}(m))$$

(cfr. [F] ch. III proof of prop. 31.1).

THEOREM (3.3). *The map*

$$h_0 : \mathcal{O}(C) \to H^1(\mathcal{U}, \mathcal{O}(-2))$$

is an isomorphism of topological vector spaces.

Proof. The map h_0 can be factorised as $h_0 = \hat{j} \circ l_0$ trought the map $l_0 : \mathcal{O}(C) \to$ $\to F_{-2}$ defined by:

$$l_0(k)(\omega, \pi) = \frac{1}{\pi_0 \pi_1} \cdot k\left(\frac{-i\omega_0}{\pi_0}, \frac{-i\omega_0}{\pi_1}, \frac{-i\omega_1}{\pi_0}, \frac{-i\omega_1}{\pi_1} \right).$$

The function $l_0(k)$ is in F_{-2} since it is holomorphic in $\widetilde{U}$, homogeneous of degree -2 and it has:

$$\lim_{\pi_0 \to \infty} (k)(\omega, \pi) = 0, \qquad \lim_{\pi_1 \to \infty} (k)(\omega, \pi) = 0.$$

If $l_0(k) = 0$ then k is zero on a dense open subset of C and therefore is zero.

Taken $f = \sum_{l_0, l_1} \dfrac{A_l(\omega)}{\pi_0^{l_0+1} \cdot \pi_1^{l_1+1}}$ in F_{-2} (with $\deg(A_l) = l_0 + l_1$) let's consider the function

$$k(z) = -z_{00} \cdot z_{01} \cdot f(-z_{00} z_{01}, -z_{01} z_{10}, i z_{01}, i z_{00}) =$$

$$= \sum_l \left(\frac{z_{01}}{z_{00}}\right)^{l_1} \cdot A_l(iz_{00}, iz_{10}) = \sum_l \left(\frac{z_{11}}{z_{01}}\right)^{l_1} \cdot A_l(iz_{00}, iz_{10}) =$$

$$= \sum_l \left(\frac{z_{00}}{z_{01}}\right)^{l_0} \cdot A_l(iz_{01}, iz_{11}) = \sum_l \left(\frac{z_{10}}{z_{11}}\right)^{l_0} \cdot A_l(iz_{01}, iz_{11})$$

this function is holomorphic on everyone of the four open subsets of C:

$$U_{jk} = \{z \in C : z_{jk} \neq 0\} \qquad (j, k = 0, 1)$$

therefore k is holomorphic on $C - \{0\}$.

Since C is a perfect space, k can be extended to all C (cfr. [BS] cor. 3.12 page 79), it is easy now to check that $l_0(k) = f$.

This proves that l_0 is a continuous isomorphism of Fréchet spaces (then a topological isomorphism). ∎

THEOREM (3.4). *The diagram:*

$$\begin{array}{ccc} & \overset{\mathscr{P}_0}{\longleftarrow} & H^1(\mathscr{U}, \mathscr{O}(-2)) \\ \mathscr{Z}_0(\mathbb{C}^4) & & \uparrow h_0 \\ & \underset{\tau}{\longrightarrow} & \mathscr{O}(C) \end{array}$$

«commutes».

In particular the map $\mathscr{P}_0$ is an isomorphism and

$$\mathscr{P}_0^{-1} = h_0 \circ \tau.$$

Therefore for every φ in $\mathscr{Z}_0(\mathbb{C}^4)$ the function defined for (ω, π) in $\mathbb{C}^2 \times \mathbb{C}^{*2}$ by:

$$\mathscr{P}_0^{-1}(\varphi)(\omega, \pi) =$$

$$\frac{1}{\pi_0 \cdot \pi_1} \cdot \frac{1}{(2\pi i)^2} \cdot \oint\!\!\oint \frac{1}{\nu_0 \nu_1} \cdot \varphi\left(\frac{-i\omega_0}{\nu_0}, \frac{-i\omega_0}{\nu_1}, \frac{-i\omega_1}{\nu_0}, \frac{-i\omega_1}{\nu_1}\right) \cdot \chi\left(\frac{\nu_0}{\pi_0}, \frac{\nu_1}{\pi_1}\right) d\nu$$

is an inverse twistor function for φ.

Proof. We have to prove that $\mathscr{P}_0 \circ h_0 \circ \tau = \mathrm{id}$; since $\mathscr{X}_0(\mathbb{C}^4)$ is «generated» by the polynomials Q_m and all the maps are continuous is enough to prove $(\mathscr{P}_0 \circ \circ h_0 \circ \tau)(Q_m) = Q_m$ for every m in M.

Infact

$$(\mathscr{P}_0 \circ h_0 \circ \tau)(Q_m) = (\mathscr{P}_0 \circ h_0)(z^{(m)}) =$$

$$= \mathscr{P}_0\left(\frac{1}{i^{|m|/2}} \cdot \frac{\omega_0^{m'_0} \cdot \omega_1^{m'_1}}{\pi_0^{m''_0+1} \cdot \pi_1^{m''_1+1}}\right) = \frac{1}{i^{|m|/2}} \cdot i^{|m|/2} \cdot Q_m. \qquad \blacksquare$$

It is very important to observe that in the formulas of theorems (3.2) and (3.4) what really is used of the function φ is its restriction to the cone C.

If $\varphi(x) = \varphi(0, x_{01}, 0, x_{11})$ then

$$\mathscr{P}_0^{-1}(\varphi)(\omega, \pi) = \frac{1}{\pi_0 \cdot \pi_1} \cdot \varphi\left(0, \frac{-i\omega_0}{\pi_1}, 0, \frac{-i\omega_1}{\pi_1}\right)$$

and if $\varphi(x) = \varphi(x_{00}, 0, x_{10}, 0)$ then

$$\mathscr{P}_0^{-1}(\varphi)(\omega, \pi) = \frac{1}{\pi_0 \cdot \pi_1} \cdot \varphi\left(\frac{-i\omega_0}{\pi_0}, 0, \frac{-i\omega_1}{\pi_0}, 0\right).$$

§4. THE SPACES $\mathscr{X}_n$ OF MASSLESS FREE SPINOR FIELDS

A massless free field of helicity $n/2$ on $\mathbb{C}^4$ (for $n \geqslant 1$) is an $(n+1)$-ple of function $(\varphi_{p_0,p_1})_{|p|=n}$ holomorphic on $\mathbb{C}^4$ verifying the differential equations (Dirac equation):

$$\begin{cases} \dfrac{\partial \varphi_{p_0,p_1}}{\partial x_{00}} = \dfrac{\partial \varphi_{p_0+1,p_1-1}}{\partial x_{01}} \\ \qquad\qquad\qquad\qquad\qquad \text{(for } p_0 = 0, \ldots, n-1) \\ \dfrac{\partial \varphi_{p_0,p_1}}{\partial x_{10}} = \dfrac{\partial \varphi_{p_0+1,p_1-1}}{\partial x_{11}} \end{cases}$$

$$\mathscr{X}_n(\mathbb{C}^4) = \{(\varphi_p)_{|p|=n} : \text{massless free fields on } \mathbb{C}^4 \text{ of helicity } n/2\}.$$

$\mathscr{X}_n(\mathbb{C}^4) \subset \mathscr{X}_0^{n+1}(\mathbb{C}^4)$, that is $\square\, \varphi_p = 0$ for every component φ_p of a field of helicity $n/2$.

For every k in M with $k'_1 \geqslant n$ let:

$$Q_{k,p} = Q_{k,p_0,p_1} = \begin{cases} Q_{k_0',k_1'-n,k_0''-p_0,k_1''-p_1} & (p_0 \leqslant k_0'', p_1 \leqslant k_1'') \\ 0 \text{ if } p_0 > k_0'' \text{ or } p_1 > k_1''. \end{cases}$$

Set $M^{(n)} = \{k \in M : k_1' \geqslant n\}$.

It is an easy computation to prove that for every $k \in M^{(n)}$ the $(n+1)$-uple $(Q_{k,p})$ is in $\mathscr{X}_n(\mathbb{C}^4)$. For every $k \in M^{(n)}$ we will call an elementary field the $(n+1)$-uple $(Q_{k,p})_p = Q_{k,\cdot}$.

Set: $$I_n = \{p = (p_0, p_1) : |p| = n\}.$$

For (k,p) and (h,q) in $M \times I_n$ we will say that (k,p) and (h,q) are equivalent if:

$$k_0' = h_0', \quad k_1' = h_1', \quad k_0'' + p_0 = h_0'' + q_0, \quad k_1'' + p_1 = h_1'' + q_1.$$

The map $t : M \times I_n \to M^{(n)}$ defined by:

$$t(k,p) = (k_0', k_1' + n, k_0'' + p_0, k_1'' + p_1)$$

has the property that $t(k,p) = t(h,q)$ if and only if $(k,p) \sim (h,q)$ and is surjective.

For every $\varphi = (\varphi_p)$ in $\mathscr{X}_0^{n+1}(\mathbb{C}^4)$ we will write:

$$\varphi_p = \sum_k a_k(\varphi_p) \cdot Q_k .$$

THEOREM (4.1). *The $(n+1)$-uple $(\varphi_p)_p \in \mathscr{X}_0^{n+1}(\mathbb{C}^4)$ is in $\mathscr{X}_n(\mathbb{C}^4)$ if and only if: $a_k(\varphi_p) = a_h(\varphi_q)$ whenever $(k,p) \sim (h,q)$.*

Proof. ($\Rightarrow$) If $(k,p) \sim (h,q)$ and $k_0'' = h_0''$ then $p = q$ and $k = h$. Let's suppose $h_0'' < k_0''$ and set $b = k_0'' - h_0'' = q_0 - p_0 = p_1 - q_1$, we can go from (k,p) to (h,q) with b steps following the chain $(k^{(0)}, p^{(0)}), \ldots, (k^{(b)}, p^{(b)})$, where for each $r = 0, \ldots, b$ we set: $(k^{(r)}, p^{(r)}) = ((k_0', k_1', k_0'' - r, k_1'' + r), (p_0 + r, p_1 - r))$.

If $k_0' > 0$ then from $\dfrac{\partial \varphi_{(p_0,p_1)}}{\partial x_{00}} = \dfrac{\partial \varphi_{(p_0+1,p_1-1)}}{\partial x_{01}}$ if follows $a_{(l_0'+1,l_1',l_0''+1,l_1'')}(\varphi_{p_0,p_1}) = a_{(l_0'+1,l_1',l_0'',l_1''+1)}(\varphi_{(p_0+1,p_1-1)})$, then $a_k(r)(\varphi_p(r)) = a_k(r+1)(\varphi_p(r+1))$ (taking $l_0' = k_0' - 1$, $l_1' = k_1'$, $l_0'' = k_0'' - r - 1$, $l_0'' = k_0'' + r$) and therefore $a_k(\varphi_p) = a_h(\varphi_q)$.

If $k_0' = 0$ since $k_1' > 0$ the conclusion follows from $\dfrac{\partial \varphi_{(p_0,p_1)}}{\partial x_{10}} = \dfrac{\partial \varphi_{(p_0+1,p_1-1)}}{\partial x_{11}}$ in an analogous way.

($\Leftarrow$) Since $((l_0'+1, l_1', l_0''+1, l_1''), (p_0, p_1)) \sim ((l_0'+1, l_1', l_0'', l_1''+1), (p_0+1, p_1-1)$

and $((l'_0, l'_1 + 1, l''_0 + 1, l''_1), (p_0, p_1)) \sim ((l'_0, l'_1 + 1, l''_0, l''_1 + 1), (p_0 + 1, p_1 - 1))$, set $v = (1, -1), u_{00} = (1, 0, 1, 0), u_{01} = (1, 0, 0, 1), u_{10} = (0, 1, 1, 0)$ and $u_{11} = (0, 1, 0, 1)$ we have:

$$\frac{\partial \varphi_p}{\partial x_{00}} = \sum_l a_{l + u_{00}}(\varphi_p) \cdot (l'_0 + 1) \cdot Q_l =$$

$$= \sum_l a_{l + u_{01}}(\varphi_{p+v}) \cdot (l'_0 + 1) \cdot Q_l = \frac{\partial \varphi_{p \mp v}}{\partial x_{01}}$$

and

$$\frac{\partial \varphi_p}{\partial x_{10}} = \sum_l a_{l + u_{10}}(\varphi_p) \cdot (l'_1 + 1) \cdot Q_l =$$

$$= \sum_l a_{l + u_{11}}(\varphi_{p+v}) \cdot (l'_1 + 1) \cdot Q_l = \frac{\partial \varphi_{p+v}}{\partial x_{11}} . \qquad \blacksquare$$

THEOREM (4.2). *The* $(n + 1)$*-uple* (φ_p) *of* $\mathscr{Y}_0^{n+1}(\mathbb{C}^4)$ *is in* $\mathscr{Y}_n(\mathbb{C}^4)$ *if and only if there exist coefficients* $(\alpha_m)_{m \in M^{(n)}}$ *such that:*

$$\varphi_. = \sum_m \alpha_m \cdot Q_{m,.}$$

and

$$\lim_{d \to \infty} \sqrt[d]{|\alpha|^{(d)}} = 0.$$

In this case the expression is unique.

Proof. If $\varphi_. = \sum_m \alpha_m \cdot Q_{m,.}$ then for the continuity of the partial derivatives the Dirac equations are verified for $\varphi_.$ as for the $Q_{m,.}$.

Viceversa if $\varphi_.$ is in $\mathscr{Y}_n(\mathbb{C}^4)$ we know for the preceding theorem that $a_k(\varphi_p) = a_h(\varphi_q)$ whenever $(k, p) \sim (h, q)$; therefore for every m in $M^{(n)}$ we can define $\alpha_m = a_k(\varphi_p)$ for whatever (k, p) in $t^{-1}(m)$.

We have:

$$\sum_{m \in M^{(n)}} \alpha_m \cdot Q_{m,p} = \sum_{\substack{m \in M^{(n)} \\ m'' \geq p}} \alpha_m \cdot Q_{m'_0, m'_1 - n, m''_0 - p_0, m''_1 - p_1} =$$

$$= \sum_k a_k(\varphi_p) \cdot Q_k = \varphi_p .$$

Since $|\alpha|^{(d)} \leqslant \sum_p |a(\varphi_p)|^{(d-2\cdot n)}$ it follows from $\lim_{d\to\infty} \sqrt[d]{|a(\varphi_p)^{(d)}|} = 0$ that $\lim_{d\to\infty} \sqrt[d]{|\alpha|^{(d)}} = 0.$

If $\varphi_. = \sum_m \alpha_m \cdot Q_{m,.} = \sum_m \beta_m \cdot Q_{m,.}$, taken m in $M^{(n)}$ we can find p such that $m'' \geqslant p$; then:

$$0 = \varphi_p - \varphi_p = \sum_{\substack{m \\ m'' \geqslant p}} (\alpha_m - \beta_m) \cdot Q_{m,p} \text{ implies } \alpha_m = \beta_m . \qquad \blacksquare$$

§5. THE SPACES $\mathscr{X}_n$ AS MODULES ON $\mathscr{X}_0$

For every $\gamma = \sum_u a_u \cdot Q_u$ in $\mathscr{X}_0(\mathbb{C}^4)$ and every $\varphi_. = \sum_{v\in M^{(n)}} b_v \cdot Q_{v,.}$ in $\mathscr{X}_n(\mathbb{C}^4)$ we define the product $\gamma * \varphi_.$ as the $(n+1)$-uple in $\mathscr{X}_n(\mathbb{C}^4)$:

$$\gamma * \varphi_. = \sum_{m\in M^{(n)}} \left(\sum_{u+v=m} a_u \cdot b_v \right) \cdot Q_{m,.}.$$

The product is well defined: the series of $\gamma * \varphi_.$ is convergent on $\mathbb{C}^4$ as the series $\sum_m \sum_{u+v=m} |a_u| \cdot |b_v| \cdot y^m$ product of the two convergent series $\sum_u |a_u| \cdot y^u$ and $\sum_v |b_v| \cdot y^v$.

It is not difficult to verify that $\mathscr{X}_n$ with this product on the scalar fields becomes a module on $\mathscr{X}_0$.

Let's denote by $\mathfrak{J}$ the ideal generated in $\mathscr{X}_0(\mathbb{C}^4)$ by the elements x_{10} and x_{11}. Obviously we will denote by $\mathfrak{J}^n$ the n-th power of $\mathfrak{J}$.

The ideal $\mathfrak{J}^n$ is «generated» by the polynomials Q_m with $m \in M^{(n)}$.

THEOREM (5.1). *The map $\epsilon_n : \mathscr{X}_n(\mathbb{C}^4) \to \mathfrak{J}^n$ defined by:*

$$\epsilon_n\left(\sum_{m\in M^{(n)}} a_m Q_{m,.} \right) = \sum_{m\in M^{(n)}} a_m \cdot Q_m$$

is an isomorphism of $\mathscr{X}_0$-modules.

For every function $\varphi \in \mathcal{O}(\mathbb{C}^4)$ let's denote by $\varphi^{(0)}$ and $\varphi^{(1)}$ the functions defined by:

$$\varphi^{(0)}(x) = \varphi(0, x_{01}, 0, x_{11}) \quad \text{and} \quad \varphi^{(1)}(x) = \varphi(x_{00}, 0, x_{10}, 0)$$

for every $x \in \mathbb{C}^4$. The functions $\varphi^{(0)}$ and $\varphi^{(1)}$ are in $\mathscr{X}_0(\mathbb{C}^4)$.

THEOREM (5.2.) *For every $\varphi_.$ in $\mathscr{X}_n(\mathbb{C}^4)$ it holds:*

$$\epsilon_n(\varphi_.) = \frac{1}{2} \cdot \Bigg[x_{11}^{*n} * (\varphi_{0n} + \varphi_{0n}^{(0)}) + x_{10}^{*n} * (\varphi_{n0} + \varphi_{n0}^{(1)}) + + \sum_{\substack{|p|=n \\ p_0, p_1 > 0}} x_{10}^{*p_0} * x_{11}^{*p_1} * (\varphi_{p_0 p_1}^{(0)} + \varphi_{p_0 p_1}^{(1)}) \Bigg].$$

Proof. Observe first that:

$$Q_{m,p}^{(0)} = \begin{cases} Q_{m,p} & \text{if } m_0'' = p_0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$Q_{m,p}^{(1)} = \begin{cases} Q_{m,p} & \text{if } m_1'' = p_1 \\ 0 & \text{otherwise} \end{cases}$$

In all possible cases (there are sixteen) the expression on the right gives, for $\varphi_. = Q_{m,.}$, the polynomial Q_m as ϵ_n does on $Q_{m,.}$. For the continuity of ϵ_n and of the expression the theorem holds for every $\varphi_.$ in $\mathscr{X}_n$. ∎

If we set:

$$\psi_{0n} = \frac{1}{2} \cdot [\varphi_{0n} + \varphi_{0n}^{(0)}]$$

$$\psi_{n0} = \frac{1}{2} \cdot [\varphi_{n0} + \varphi_{n0}^{(1)}]$$

$$\psi_p = \frac{1}{2} \cdot [\varphi_p^{(0)} + \varphi_p^{(1)}] \qquad (\text{for } p_0, p_1 > 0 \text{ and } |p| = n)$$

we have $\epsilon_n(\varphi_.) = \sum_{|p|=n} x_{10}^{*p_0} * x_{11}^{*p_1} * \psi_{p_0 p_1}$.

Let's denote by S the plane in the cone C defined by:

$$S = \{z \in \mathbb{C}^4 : z_{10} = z_{11} = 0\}.$$

The map $\tau : \mathscr{Z}_0(\mathbb{C}^4) \to \mathscr{O}(C)$ translates $\mathfrak{J}$ in the ideal $\mathscr{T}_S(C)$. Therefore $\tau_n = \tau \circ \epsilon_n : \mathscr{Z}_n(\mathbb{C}^4) \to \mathscr{T}^n_S(C)$ is an isomorphism between a $\mathscr{Z}_0$-module and a $\mathscr{O}(C)$-module.

Observe that $\tau_n(\varphi) = \sum_p z_{10}^{p_0} \cdot z_{11}^{p_1} \cdot \tau(\psi_p)$. Let's consider the map:

$$h_n : \mathscr{T}^n_S(C) \to H^1(\mathbb{P}^3 - L, \mathscr{O}(-n-2))$$

defined by:

$$h_n(k) = \left[\frac{1}{\pi_0\pi_1} \cdot \frac{1}{(-i\omega_1)^n} \cdot k\left(\frac{-i\omega_0}{\pi_0}, \frac{-i\omega_0}{\pi_1}, \frac{-i\omega_1}{\pi_0}, \frac{-i\omega_1}{\pi_1}\right)\right].$$

Since for k in $\mathscr{T}^n_S(C)$ we have:

$$k(z) = \sum_{|p|=n} z_{10}^{p_0} \cdot z_{11}^{p_1} \cdot k_p(z)$$

then $k\left(\frac{-i\omega_0}{\pi_0}, \frac{-i\omega_0}{\pi_1}, \frac{-i\omega_1}{\pi_0}, \frac{-i\omega_1}{\pi_1}\right)$ is divisible by ω_1^n and the function in the definition of $h_n(k)$ is well defined for every ω and π with $\pi_0 \neq 0$ and $\pi_1 \neq 0$ and homogeneous of degree $-n-2$.

THEOREM (5.3). *The map:*

$$h_n : \mathscr{T}^n_S(C) \to H^1(\mathbb{P}^3 - L, \mathscr{O}(-n-2))$$

is an isomorphism of topological vector spaces.

Proof. The map h_n can be factorised as $h_n = \hat{j} \circ l_n$ trought the maps:

$$\hat{j} : F_{-n-2} \to H^1(\mathscr{U}, \mathscr{O}(-n-2)) \text{ and } l_n : \mathscr{T}^n_S(C) \to F_{-n-2}$$

where:

$$l_n(k)(\omega, \pi) = \frac{1}{\pi_0\pi_1} \cdot \frac{1}{(-i\omega_1)^n} \cdot k\left(\frac{-i\omega_0}{\pi_0}, \frac{-i\omega_0}{\pi_1}, \frac{-i\omega_1}{\pi_0}, \frac{-i\omega_1}{\pi_1}\right).$$

The function $l_n(k)$ is in F_{-n-2} since $\lim_{\pi_0 \to \infty} l_n(k) = 0$ and $\lim_{\pi_1 \to \infty} l_n(k) = 0$.

If $l_n(k) = 0$ then $k = 0$ an a dense subset of C. For every $f \in F_{-n-2}$ we can find functions $\{f_{p_0,p_1}\}_{p_0+p_1=n}$ in F_{-2} such that $f = \sum_{p_0+p_1=n} \frac{1}{\pi_0^{p_0}\pi_1^{p_1}} \cdot f_{p_0 p_1}$ infact:

$$f = \sum_{\substack{k,l \\ |l|=|k|+n}} a_{kl} \cdot \frac{\omega_0^{k_0}\omega_1^{k_1}}{\pi_0^{l_0+1}\pi_1^{l_1+1}} = \sum_{\substack{k,h,p \\ |k|=|h| \\ |p|=n}} a_{k,h+p} \cdot \frac{\omega_0^{k_0}\omega_1^{k_1}}{\pi_0^{h_0+p_0+1}\pi_1^{h_1+p_1+1}} =$$

$$= \sum_{p_0+p_1=n} \frac{1}{\pi_0^{p_0}\pi_1^{p_1}} \cdot \left(\sum_{\substack{k,h \\ |k|=|h|}} a_{k,h+p} \cdot \frac{\omega_0^{k_0}\omega_1^{k_1}}{\pi_0^{h_0+1}\pi_1^{h_1+1}} \right).$$

Proceeding as in the proof of theorem (3.3) we can find functions $\{k_{p_0,p_1}\}_{p_0+p_1=n}$ in $\mathcal{O}(C)$ such that:

$$f_p(\omega,\pi) = \frac{1}{\pi_0\pi_1} \cdot k_p\left(\frac{-i\omega_0}{\pi_0}, \frac{-i\omega_0}{\pi_1}, \frac{-i\omega_1}{\pi_0}, \frac{-i\omega_1}{\pi_1}\right).$$

It is easy now to prove that $l_n(\sum_{p_0+p_1=n} z_{10}^{p_0} \cdot z_{11}^{p_1} \cdot k_{p_0p_1}(z)) = f$.

Therefore l_n is a continuous isomorphism among Fréchet spaces and then a topological ismorphism. The same holds for h_n. ■

THEOREM (5.4). *The diagram:*

$$\begin{array}{ccc} & \xleftarrow{\;\mathcal{P}_n\;} & H^1(\mathbb{P}^3 - L, \mathcal{O}(-n-2)) \\ \mathcal{Z}_n(\mathbb{C}^4) & & \uparrow h_n \\ & \xrightarrow{\;\tau_n\;} & \mathcal{T}_S^n(C) \end{array}$$

«commutes»; in particular $\mathcal{P}_n$ is an isomorphism and $\mathcal{P}_n^{-1} = h_n \circ \tau_n$. Therefore taken $\varphi_.$ in $\mathcal{Z}_n(\mathbb{C}^4)$ the function defined for every (ω,π) in $\mathbb{C}^2 \times \mathbb{C}^{*2}$:

$$\mathcal{P}_n^{-1}(\varphi_.)(\omega,\pi) =$$

$$= \sum_{p_0+p_1=n} \frac{1}{\pi_0^{p_0+1}\pi_1^{p_1+1}} \cdot \frac{1}{(2\pi i)^2} \cdot \oint\oint \frac{1}{\nu_0 \cdot \nu_1} \cdot \psi_p\left(\frac{-i\omega_0}{\nu_0}, \frac{-i\omega_0}{\nu_1}, \frac{-i\omega_1}{\nu_0}, \frac{-i\omega_1}{\nu_1}\right) \cdot$$

$$\cdot \chi\left(\frac{\nu_0}{\pi_0}, \frac{\nu_1}{\pi_1}\right) d\nu_0 \cdot d\nu_1 = \sum_{p_0+p_1=n} \frac{1}{\pi_0^{p_0} \cdot \pi_1^{p_1}} \cdot \mathcal{P}_0^{-1}(\psi_p)(\omega,\pi)$$

is an inverse twistor function for $\varphi_.$.

(The integrals are taken on circles $|\nu_j| = \epsilon \cdot |\pi_j|$ for small $\epsilon > 0$).

Proof. We have to prove that $(\mathcal{P}_n \circ h_n \circ \tau_n)(Q_m) = Q_m$ for every $m \in M^{(n)}$.

This follows from:

$$h_n(z^{(m)}) = \frac{(-1)^n}{i^{|m|/2+n}} \cdot \frac{\omega_0^{m'_0}}{\pi_0^{m''+1}} \cdot \frac{\omega_1^{m'_1-n}}{\pi_1^{m''+1}}$$

and:

$$\mathscr{P}_n\left(\frac{\omega_0^{m'_0} \cdot \omega_1^{m'_1-n}}{\pi_0^{m''_0+1}\pi_1^{m''_1+1}}\right) = \mathscr{P}_0\left(\frac{\omega_0^{m'_0} \cdot \omega_1^{m'_1-n}}{\pi_0^{m''_0-p_0+1} \cdot \pi_1^{m''_1-p_1+1}}\right)_p =$$

$$= i^{\frac{|m|-2\cdot n}{2}} \cdot Q_{m'_0, m'_1-n, m''_0-p_0, m''_1-p_1}. \qquad \blacksquare$$

Remembering that:

$$\mathscr{P}_0^{-1}(\varphi^{(0)})(\omega, \pi) = \frac{1}{\pi_0 \cdot \pi_1} \cdot \varphi(\alpha^{(0)}(\omega, \pi))$$

and

$$\mathscr{P}_0^{-1}(\varphi^{(1)})(\omega, \pi) = \frac{1}{\pi_0 \cdot \pi_1} \cdot \varphi(\alpha^{(1)}(\omega, \pi))$$

where:

$$\alpha^{(0)}(\omega, \pi) = \left(0, \frac{-i\omega_0}{\pi_1}, 0, \frac{-i\omega_1}{\pi_1}\right)$$

and

$$\alpha^{(1)}(\omega, \pi) = \left(\frac{-i\omega_0}{\pi_0}, 0, \frac{-i\omega_1}{\pi_0}\right), 0$$

we can write the formula in theorem (5.4) in a simpler form:

$$\mathscr{P}_n^{-1}(\varphi)(\omega, \pi) = \frac{1}{2} \cdot \left[\frac{1}{\pi_1^n} \cdot \mathscr{P}_0^{-1}(\varphi_{0n})(\omega, \pi) + \frac{1}{\pi_0^n} \cdot \mathscr{P}_0^{-1}(\varphi_{n0})(\omega, \pi) + \right.$$

$$+ \frac{1}{\pi_0 \cdot \pi_1^{n+1}} \cdot \varphi_{0n}(\alpha^{(0)}(\omega, \pi)) + \frac{1}{\pi_0^{n+1}\pi_1} \cdot \varphi_{n0}(\alpha^{(1)}(\omega, \pi)) +$$

$$\left. + \sum_{\substack{p_0+p_1=n \\ p_0, p_1 > 0}} \frac{1}{\pi_0^{p_0+1} \cdot \pi_1^{p_1+1}} \cdot (\varphi_p(\alpha^{(0)}(\omega, \pi)) + \varphi_p(\alpha^{(1)}(\omega, \pi))\right].$$

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